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On Deferred Statistical Convergence in Fuzzy Normed Linear Spaces

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Abstract

In this paper, we introduce the notions of deferred statistically convergent sequence and deferred statistically Cauchy sequence in a fuzzy normed linear space(FNS) and establish some basic facts. We also define deferred statistical limit points and deferred statistical cluster points of a sequence in FNS and investigate the relations between these concepts.

Keywords: Statistical cluster point, statistical convergence, fuzzy normed linear space, statistical limit point, fuzzy number.

1|Introduction

The concept of a "fuzzy norm" was initially introduced by Katsaras [32] while exploring "fuzzy topological vector spaces". In 1992, Felbin[20] introduced the concept of a "fuzzy norm" on a linear space, building upon the notion of fuzzy numbers. This concept was rooted in the treatment of a "fuzzy metric" originally proposed by Kaleva and Seikkala[31]. Subsequent studies, such as those in [12, 15], delved into various topological properties of these "fuzzy normed linear spaces" (FNSs). Additionally, different types of FNSs were explored in works like [4, 8]. In this article, we adopt the methodology outlined in Felbin's[20] work. In the study of FNSs, the convergence of sequences of "fuzzy numbers" is a key method for defining ordinary convergence within these spaces. This paper seeks to utilize the concept of "statistical convergence" of fuzzy number sequences to explore a more extensive form of convergence, namely "statistical convergence" within an FNS. The objective is to establish foundational

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principles and essential insights in this context. The objective is to establish foundational principles and essential insights in this contex, to more “fuzzy set” [2, 14, 33, 50, 53, 57].

In 1951, Fast [19] first introduced statistical convergence as a more general form of convergence for real sequences. This concept has since been extensively studied and applied by numerous researchers [5, 6, 7, 21, 47, 48, 51]. In 1993, Fridy introduced the notions of “statistical cluster points” and “statistical limit points” of real sequences [22], with applications in turnpike theory, especially in the study of optimal paths [45]. Additionally, “statistical convergence” has been studied in broader abstract spaces like “fuzzy number” spaces [1, 40] , “locally convex spaces” [39] and “Banach spaces” [10, 34, 44].

In 1932, Agnew [3] introduced the idea of the “deferred Cesàro mean” for real or complex number sequences. Later, in 2016, Küçükaslan and Yılmaztürk [35] built upon this concept by presenting the notion of deferred statistical convergence. Gupta and Bhardwaj [26] further advanced this idea by incorporating modulus functions, which led to the development of “strongly deferred Cesàro” convergence and its connection with “deferred f-statistical convergence”. Additional insights and studies on deferred statistical convergence can be found in [11, 13, 16, 17, 24, 25, 28, 29, 30, 36, 37, 38, 41, 42, 43, 49, 56]. Given the importance of sequence convergence in FNSs for fuzzy functional analysis, it is believed that extending the concept of statistical convergence to FNSs will significantly enhance this field.

As a preparatory step, this paper is organized into several sections to systematically present our research findings. In the second section, we provide essential groundwork by presenting “preliminary definitions” and results concerning “fuzzy numbers”, “fuzzy normed linear spaces” (FNSs) and “statistical convergence”. This sets the stage for a comprehensive understanding of the concepts under investigation.

Moving to Section 3, we introduce novel notions within the context of FNSs, specifically focusing on deferred statistical convergence and deferred statistically Cauchy sequences. We derive key results in this section, contributing to the foundational aspects of deferred statistical convergence theory in the context of “fuzzy normed linear spaces”.

In the subsequent section, we delve into the definition of “deferred statistical limit points” and “deferred statistical cluster points” for sequences within an FNS. Our exploration includes an in-depth investigation into the relationships between these newly introduced concepts. Finally, we discuss potential application areas of this study and offer insights into future directions for research.

2|Definitions and Preliminaries

In this portion, we begin by some necessary definitions concerning “fuzzy numbers”, FNSs and “deferred statistical convergence”:

Definition 2.1 ([46])**.** Let $\mu : \mathbb{R} \to [0, 1]$ represent a fuzzy subset of the real numbers, $\mathbb{R}$. Then, for any κ belonging to the interval $[0, 1]$, the κ-level set of μ is represented by $\mu\kappa$ and defined as the set measure μ defined on the real numbers $\mathbb{R}$, the notation $[\mu]$ denotes the set of points t in $\mathbb{R}$ where the measure μ evaluates to at least κ, when $0 < \kappa \leqslant 1$. If $\kappa = 0$, $[\mu]$ refers to the closure of the set of points t in $\mathbb{R}$ where μ evaluates to strictly greater than 0.

Definition 2.2 ([46])**.** A fuzzy set denoted by μ defined on $\mathbb{R}$ is termed a “fuzzy number” subject to the specified conditions:

(1) μ is normal, meaning $\exists$ a specific point t_0 in $\mathbb{R}$ where μ attains its maximum membership grade of 1.

(2) μ is fuzzy convex, indicating that for any pair of real numbers t_1 and t_2 and any λ in the interval $[0, 1]$, $\mu(\lambda t_1 + (1 - \lambda)t_2)$ is greater than or equal to the minimum of $\mu(t_1)$ and $\mu(t_1)$.

(3) μ exhibits upper semi-continuous.

(4) The set $[\mu]_0$, consisting of all t in $\mathbb{R}$ where $\mu(t)$ is greater than 0, is compact.

Here r(a “real number”) can be represented and defined as follows, “fuzzy number” $r^{\tilde{}}$, if t equals r, then $r^{\tilde{}}$ (t) equals 1, if t is not equal to r, then $r^{\tilde{}}(t)$ equals 0. If each κ-level set $[\mu]_{\kappa}$ forms a non-empty, bounded, and closed interval, denoted as $[\mu]_{\kappa} = [\mu_{\kappa}{}^{-}, \mu_{\kappa}{}^{+}]$, then μ qualifies as a fuzzy number.

Definition 2.3 ([46])**.** Consider $\mathcal{L}(\mathbb{R})$, the collection encompassing all "fuzzy numbers". If a fuzzy number μ is a member of $\mathcal{L}(\mathbb{R})$ and its membership grade $\mu(t)$ is zero for t less than zero, it is referred to as a "non-negative fuzzy number".

The set $\mathcal{L}^*(\mathbb{R})$ denotes the group of all "non-negative fuzzy numbers". We can assert that $\mu \in \mathcal{L}^*(\mathbb{R})$ iff $\mu^- \geqslant 0$ for each $\kappa \in [0,1]$. Clearly, $\widetilde{0} \in \mathcal{L}^*(\mathbb{R})$.

$\preccurlyeq$ ("a partial order") on $\mathcal{L}(\mathbb{R})$ is represented by

$$\mu \preccurlyeq \nu \text{ iff } \mu^- \leqslant \nu^- \quad \text{and} \quad \mu^+ \leqslant \nu^+ \text{ for all } \kappa \in [0,1].$$

The inequality is represented by $\prec$ on $\mathcal{L}(\mathbb{R})$ is established as $\mu \prec \nu$ (or $\nu \succ \mu$) iff $\mu^- < \nu^-$ and $\mu^+ < \nu^+$ for all $\kappa \in [0,1]$.

Definition 2.4 ([46])**.** We define the operations of "addition"($\oplus$), multiplication($\otimes$) and "scalar multiplication" on the set $\mathcal{L}(\mathbb{R})$ as follows:

(i) The convolution of two functions μ and ν, denoted as $(\mu \oplus \nu)(t)$ is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by shifting and overlapping the functions μ and ν.

(ii) The product convolution of two functions μ and ν, denoted as $(\mu \otimes \nu)(t)$ is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by scaling and overlapping the functions μ and ν.

(iii) Scalar multiplication of a function μ by a scalar k is defined for any t in the real numbers $\mathbb{R}$ as μ evaluated at t/k, where k is a real number not equal to zero. Additionally when $k = 0$, the result is the zero function $\widetilde{0}(t)$.

Theorem 2.1 ([46])**.** Let μ, ν belongs to $\mathcal{L}(\mathsf{R})$ and κ belongs to [0, 1]. Consequently,

$$(i)\ [\mu \oplus \nu]_\kappa = \left[\mu_\kappa^- + \nu_\kappa^-, \mu_\kappa^+ + \nu_\kappa^+\right]$$
$$(ii)\ [\mu \otimes \nu]_\kappa = \left[\mu_\kappa^- \nu_\kappa^-, \mu_\kappa^+ \nu_\kappa^+\right] (\mu, \nu \in L^*(\mathbb{R}))$$
$$(iii)\ [k\mu]_\kappa = k[\mu]_\kappa = \begin{cases} [k\mu_\kappa^-, k\mu_\kappa^+] & \text{if } k \geqslant 0 \\ [k\mu_\kappa^+, k\mu_\kappa^-] & \text{if } k < 0. \end{cases}$$

Theorem 2.2 ([23])**.** Let μ be a fuzzy number in $\mathcal{L}(\mathbb{R})$, with κ-level sets denoted by $[\mu]_\kappa = [\mu_\kappa^-, \mu_\kappa^+]$. The theorem establishes the following:

(i) μ_κ^- is a function that is bounded, "non-decreasing" on the interval $(0,1]$ and "left-continuous",

(ii) μ_κ^+ is a function that is bounded, "non-increasing" function on $(0,1]$ and "right-continuous",

(iii) at $\kappa = 0$, both μ_0^- and μ_0^+ are continuous,

(iv) μ_1^- is less than or equal to μ_1^+.

Conversely, if functions $a(\kappa)$ and $b(\kappa)$ meet conditions (i)-(iv), $\exists$ a unique "fuzzy number" $\mu \in \mathcal{L}(\mathbb{R})$ such that $[\mu]_\kappa = [a(\kappa), b(\kappa)]$ for all $\kappa \in [0,1]$.

Definition 2.5 ([46])**.** Considering μ and ν as elements belonging to the space $\mathcal{L}(\mathbb{R})$ define the discrepancy between two measures μ and ν, represented as $\mathcal{A}(\mu,\nu)$, is defined as the supremum over all possible values of κ in the interval $[0,1]$ of the maximum absolute differences between the lower and upper variations of μ and ν. The function $\mathcal{A}$ is referred to as the supremum metric on the set $\mathcal{L}(\mathbb{R})$. If (μ_n) is a sequence in $\mathcal{L}(\mathbb{R})$ and μ is an element of $\mathcal{L}(\mathbb{R})$, we say that the sequence (μ_n) converges to μ in the metric $\mathcal{A}$, denoted as $\mu_n \xrightarrow{\mathcal{A}} \mu$ or $(\mathcal{A})\text{-}\lim_{n\to\infty} \mu_n = \mu$, if the limit as n approaches infinity of the supremum metric $\mathcal{A}(\mu_n, \mu)$ is equal to zero.

Definition 2.6 ([20])**.** Let V be a vector space in $\mathbb{R}$, equipped with a mapping $\|\cdot\| : V \to \mathcal{L}^*(\mathbb{R})$, and consider symmetric, "non-decreasing" mappings $\mathcal{L}, \mathcal{R} : [0,1] \times [0,1] \to [0,1]$, satisfying $\mathcal{L}(0,0) = 0$ and $\mathcal{R}(1,1) = 1$. This quadruple $(V, \|\cdot\|, \mathcal{L}, \mathcal{R})$ is termed an FNS, with $\|\cdot\|$ referred to as a "fuzzy norm", provided it meets the following conditions:

(i) The norm of $\mathcal{P}$ equals zero if and only if $\mathcal{P}$ is the zero vector 0.

(ii) The norm of a scalar multiple $r\mathcal{P}$ is equal to the absolute value of r multiplied by the norm of $\mathcal{P}$, for all vectors $\mathcal{P}$ in V and for all scalars r.

(iii) For any vectors $\mathcal{P}$ and $\mathcal{Q}$ in V:

(a) The norm of their sum $\mathcal{P}+\mathcal{Q}$ is greater than or equal to the minimum of their norms.

(b) The norm of their sum $\mathcal{P}+\mathcal{Q}$ is less than or equal to the maximum of their norms.

They also define functions $\mathcal{L}(\mathcal{P},\mathcal{Q})$ and $\mathcal{R}(\mathcal{P},\mathcal{Q})$ as the minimum and maximum of $\mathcal{P}$ and $\mathcal{Q}$ respectively, when $\mathcal{P}$ and $\mathcal{Q}$ are in the interval $[0,1]$. The "fuzzy normed space" is denoted as $(V,||\cdot||)$ or simply $\mathcal{P}$ when $\mathcal{L}$ and $\mathcal{R}$ follow these definitions.

Lemma 2.1 ([20]). In a normed linear space, the norm of the sum of two vectors is less than or equal to the sum of their individual norms as defined in Definition 2.6 $(iii)(a)$ (with $\mathcal{L}$ being the minimum function) is equivalent to the inequality $\|\mathcal{P}+\mathcal{Q}\|_\kappa^- \leqslant \|V\|_\kappa^- + \|\mathcal{Q}\|_\kappa^-$, holding $\mathcal{P},\mathcal{Q}\in V$ and for all $\kappa\in(0,1]$.

Lemma 2.2 ([20]). The "triangle inequality" specified in Definition 2.6(iii)(b) (with $\mathcal{R}$ being the maximum function) is equivalent to the inequality $|\mathcal{P}+\mathcal{Q}|_\kappa^+ \leqslant |V|_\kappa^+ + |\mathcal{Q}|_\kappa^+$ $\mathcal{P},\mathcal{Q}\in V$ and for all $\kappa\in(0,1]$.

Remark 2.1 ([20]). From Theorem 2.2 (iii) and Lemma 2.1, we can deduce that the condition outlined in Definition 2.6 (iii)(a) (where $\mathcal{L}$ is the minimum function) implies that

$$\lim_{\kappa\to 0}\|\mathcal{P}+\mathcal{Q}\|_\kappa^- \leqslant \lim_{\kappa\to 0}\|\mathcal{P}\|_\kappa^- + \lim_{\kappa\to 0}\|\mathcal{Q}\|_\kappa^-,$$

that is, $\|\mathcal{P}+\mathcal{Q}\|_0^- \leqslant \|\mathcal{P}\|_0^- + \|\mathcal{Q}\|_0^-$. Similarly, according to Definition 2.6 (iii)(b) (with $\mathcal{R}$ being the maximum function), it follows that the non-negative part of the sum of two elements, denoted by $\|\mathcal{P}+\mathcal{Q}\|_0^+$, is bounded above by the sum of their respective non-negative parts, $\|\mathcal{P}\|_0^+ + \|\mathcal{Q}\|_0^+$. Consequently, in a "Fuzzy Normed Space" FNS $(V,\|\cdot\|)$, the triangle inequality specified in Definition 2.6 (iii) suggests that the norm of the sum of two elements, denoted by $\|\mathcal{P}+\mathcal{Q}\|$, is less than or equal to the composition of the norms of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\|\mathcal{P}\|\oplus\|\mathcal{Q}\|$.

According to Definition 2.6, we have $\mathcal{P}$ is zero, iff $\|\mathcal{P}\|=\widetilde{0}$, iff $\|\mathcal{P}\|_\kappa^- = \|\mathcal{P}\|_\kappa^+$ is zero, for all $\kappa\in[0,1]$. Furthermore, we have $\|\mathcal{P}\|_0^-$ is greater than zero, whenever $\mathcal{P}\neq 0$. Now if r is zero,, then $[\|r\mathcal{P}\|]_\kappa = [\|\theta\|]_\kappa = [0,0] = [|r|\|\mathcal{P}\|]_\kappa$ for all $\kappa\in[0,1]$ and $\mathcal{P}\in V$. For $r\neq 0$, we have $[\|r\mathcal{P}\|]_\kappa = [|r|\|\mathcal{P}\|]_\kappa$ for each $\kappa\in[0,1]$, i.e., $\|r\mathcal{P}\|_\kappa^- = |r|\|\mathcal{P}\|_\kappa^-$and $\|r\mathcal{P}\|_\kappa^+ = |r|\|\mathcal{P}\|_\kappa^+$for each $\kappa\in[0,1]$.

Example 2.1 ([46]). Given $(V,\|\cdot\|_C)$ as a standard normed linear space, a "fuzzy norm" $\|\cdot\|$ on V can be derived as follows:

$$\|\mathcal{P}\|(t) = \begin{cases} 0 & \text{if } 0\leqslant t\leqslant \zeta\|\mathcal{P}\|_C \text{ or } t\geqslant \eta\|\mathcal{P}\|_C \\ \frac{t}{(1-\zeta)\|\mathcal{P}\|_C} - \frac{\zeta}{1-\zeta} & \text{if } \zeta\|\mathcal{P}\|_C\leqslant t\leqslant \|\mathcal{P}\|_C \\ \frac{-t}{(\eta-1)\|\mathcal{P}\|_C} + \frac{b}{\eta-1} & \text{if } \|\mathcal{P}\|_C\leqslant t\leqslant \eta\|\mathcal{P}\|_C \end{cases}$$

in the given context, $\|\mathcal{P}\|_C$ denotes the standard norm of $\mathcal{P}$ (excluding the zero vector), where $0<\zeta<1$ and $1<\eta<\infty$. For the zero vector $\mathcal{P}$ is zero,, we define $\|\mathcal{P}\|=\widetilde{0}$. Consequently, $(V,\|\cdot\|)$ constitutes a (FNS). The specific "fuzzy norm" discussed here is referred to as the triangular "fuzzy norm" [27].

Definition 2.7 ([18]). In a "fuzzy normed space" (FNS) $(V,\|\cdot\|)$, a sequence $(\mathcal{P}_r)$ converges to $\mathcal{P}\in V$ with respect to the "fuzzy norm", denoted as $\mathcal{P}_r \xrightarrow{FN} \mathcal{P}$, if the "fuzzy norm" of $\mathcal{P}_r-\mathcal{P}$ approaches zero as r tends to infinity. Specifically, this means that for any ε is greater than zero, $\exists$ a "natural number" $N(\varepsilon)$ such that for all $r\geq N(\varepsilon)$, the "fuzzy norm" $\|\mathcal{P}_r-\mathcal{P}\|$ is within ε of zero. Formally, this is expressed as:

$$\sup_{\kappa\in[0,1]}\|\mathcal{P}_r-\mathcal{P}\|_\kappa^+ = \|\mathcal{P}_r-\mathcal{P}\|_0^+ < \varepsilon.$$

Definition 2.8 ([21]). The "natural density" of a set K of "positive integers" is determined as consider counting how many elements of K are less than or equal to any large number r. Divide this count by r. As r becomes

very large, this ratio tends to a limit, which is called the natural density $\delta(K)$. For sets K with finitely many elements, the natural density $\delta(K)$ is 0.

Although the concept of natural density may not have a well-defined value for every set K, it is important to note that the upper density of K is always defined. The upper density provides a measure of the limiting the proportion of elements of a given set K about a larger set regardless of whether the natural density is defined for K or not,

$$\bar{\delta}(K) = \limsup_{r\to\infty} \frac{1}{r}|\{k \in K : k \leqslant r\}|.$$

Definition 2.9 ([40]). A sequence of "real numbers" $(\mathcal{P}_r)$ is considered to be "statistically convergent" to a "real number" a if for any positive value ε, $\exists$ a set,

$$K(\varepsilon) = \{r \in \mathbb{N} : |\mathcal{P}_r - a| \geqslant \varepsilon\}$$

has a "natural density" equal to zero. In this scenario, we denote the convergence as $st - \lim \mathcal{P}_r = a$.

Theorem 2.3 ([9]). Let $(\mathcal{P}_r)$ and $(\mathcal{Q}_r)$ be sequences of "real numbers" that converge statistically, and let $\kappa, \beta \in \mathbb{R}$. Then, the following holds:

$$\mathrm{st} - \lim\left(\kappa \mathcal{P}_r + \beta \mathcal{Q}_r\right) = \kappa \, \mathrm{st} - \lim \mathcal{P}_r + \beta \, \mathrm{st} - \lim \mathcal{Q}_r.$$

Theorem 2.4 ([52]). Let $(\mathcal{P}_r)$, $(\mathcal{Q}_r)$ and $(\mathcal{Z}_r)$ be sequences of real numbers such that $\mathcal{P}_r \leqslant \mathcal{Q}_r \leqslant \mathcal{Z}_r$, for every natural number r belonging to the set K and under the condition that the function $\delta(K)$ evaluates to 1 and $\mathrm{st} - \lim \mathcal{P}_r = \mathrm{st} - \lim \mathcal{Z}_r = a$. Then $\mathrm{st} - \lim \mathcal{Q}_r = a$.

Definition 2.10 ([40]). A sequence of "fuzzy numbers" (μ_r) is said to "statistically converge" to the "fuzzy number" μ, denoted as the statistical limit of μ_r being μ, if for every positive number ε, $\exists$ a "positive integer" N such that,

$$\delta\left(\{r \in \mathbb{N} : \mathcal{A}\left(\mu_r, \mu\right) \geqslant \varepsilon\}\right)$$

is zero.

Definition 2.11 ([3]). Consider $K \subseteq \mathbb{N}$ and let $K_{d,c}(n)$ denote the set of integers in the interval $[d(n)+1, c(n)]$ that belong to K, where sequences $d = (d(n))$ and $c = (c(n))$ consist of "non-negative integers" that satisfy the following conditions:

$$d(n) < c(n) \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} c(n) = \infty. \tag{1}$$

The "deferred density" of K is represented and defined by

$$\delta_{d,c}(K) = \lim_{n\to\infty} \tfrac{1}{c(n)-d(n)}|K_{d,c}(n)|.$$

Definition 2.12 ([35]). A sequence $(\mathcal{P}_r)$ of "real numbers" is "deferred statistically convergent" to $l \in \mathbb{R}$ if, such that

$$\lim_{n\to\infty} \frac{1}{c(n)-d(n)} \left|\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |\mathcal{P}_r - l| \geq \varepsilon\}\right|$$

is zero, for every ε, is greater than zero then $\exists$ an integer $N(\varepsilon)$ and sequences $d = (d(n))$ and $c = (c(n))$ consist of non-negative integers satisfying the conditions specified in Equation (1).

3|Main results

In this section, we explore the concepts of deferred "statistically convergent" sequences and "deferred statistically Cauchy" sequences within (FNSs). We establish fundamental properties related to these concepts and delve into the definition of deferred statistical cluster points and deferred statistical limit points for sequences in FNS. Our investigation focuses on exploring the relationships between these introduced notions, presenting key results that contribute to a deeper understanding of statistical convergence in FNSs.

Definition 3.1. Let $(V, \|\cdot\|)$ represent a FNS and $(\mathcal{P}_r)$ be a sequence in V is considered to be "deferred statistically convergent" to $\mathcal{P}$ on V, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1), then we write $\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} \mathcal{P}$, provided that $st_{d,c} - \lim \|\mathcal{P}_r - \mathcal{P}\| = \widetilde{0}$; i.e., for each ε is greater than zero, we have

$\delta_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{A}\left(\|\mathcal{P}_r - \mathcal{P}\|, \widetilde{0}\right) \geqslant \varepsilon\right\}\right)$ is zero. Or equivalently,

$$\lim_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{A}\left(\|\mathcal{P}_r - \mathcal{P}\|, \widetilde{0}\right) \geqslant \varepsilon\right\}\right|$$

is zero.

This implies that for each ε is greater than 0, and density of the set $K(\varepsilon) = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}$ is zero. That is, for every ε is greater than zero , $\|\mathcal{P}_r - \mathcal{P}\|_0^+ < \varepsilon$ for a.a.r (almost all r). The element $\mathcal{P}$ belongs to the set V serves as the statistical limit of the sequence $(\mathcal{P}_r)$.

An insightful way to understand the definition mentioned above is as follows:

$$\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} \mathcal{P} \text{ iff } st_{d,c} - \lim \|\mathcal{P}_r - \mathcal{P}\|_0^+$$

is zero.

Noting that $st_{d,c} - \lim \|\mathcal{P}_r - \mathcal{P}\|_0^+$ is zero, implies that

$$st_{d,c} - \lim \|\mathcal{P}_r - \mathcal{P}\|_\kappa^- = st_{d,c} - \lim \|\mathcal{P}_r - \mathcal{P}\|_\kappa^+$$

is zero,

for each $\kappa \in [0,1]$ since

$$0 \leqslant \|\mathcal{P}_r - \mathcal{P}\|_\kappa^- \leqslant \|\mathcal{P}_r - \mathcal{P}\|_\kappa^+ \leqslant \|\mathcal{P}_r - \mathcal{P}\|_0^+$$

holds for every n belongs to "natural number" and for each κ belongs to $[0,1]$. This conclusion follows from Theorem 2.4. Throughout this paper, when we say $(\mathcal{P}_r)$ is deferred statistically convergent to $\mathcal{P}$ belongs to V, it means that $(\mathcal{P}_r)$ is "deferred statistically convergent" to $\mathcal{P}$ belongs to V with respect to the "fuzzy norm" on V.

Remark 3.1. It is clear that,

- If $c(n)$ is n and $d(n)$ is zero, then "deferred statistical convergence" on "fuzzy normed spaces" (FNSs) coincide with the definition of statistical convergence on "fuzzy normed spaces" (FNSs) [46].

- If $c(n)$ is λ_n and $d(n)$ is zero "where λ is a non-decreasing sequence of positive integers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, λ_1 is zero", then "deferred statistical convergence" in (FNSs) coincides with "λ-statistical convergence" in (FNSs) [55].

- If $c(n) = k_n$ and $d(n) = k_{n-1}$ "for any lacunary sequence of non-negative integer satisfying $k_n - k_{n-1} \to \infty$ as $n \to \infty$", then "deferred statistical convergence" of (FNSs) coincides with "lacunary statistical convergence" of (FNSs) [54].

Example 3.1. Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ is a "normed linear space". Then the "fuzzy norm" $\|\cdot\|_F$ on $\mathbb{R}$ can be obtained as

$$\| \mathcal{P}\|(t) = \begin{cases} \dfrac{t-|\mathcal{P}|}{t+|\mathcal{P}|} & t > |\mathcal{P}| \\ 0, & t \leq |\mathcal{P}| \end{cases},$$

and $(\mathbb{R}, \|\|_F)$ is a FNS.

Example 3.2. Consider the "fuzzy norm" defined in Example 3.1 above and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). Let us define a sequence $(\mathcal{P}_r)$ as

$$\mathcal{P}_r = \begin{cases} k^2, & [\sqrt{c(n)}] - 1 < k \leq [\sqrt{c(n)}], n = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases},$$

where the value of $d(n)$ is greater than zero and less than or equal to the integer part of the square root of $c(n)$ less than one. is a "monotonic increasing sequence". Its gives, $\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} 0$.

Justification: We have $K_\varepsilon^n = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - 0\|_0^+ \geqslant \varepsilon\right\}$, For every $0 < \varepsilon < 1$, $t > |\mathcal{P}|$. This implies that,

$$\begin{aligned} K_\varepsilon^n &= \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \frac{t-|\mathcal{P}_r|}{t+|\mathcal{P}_r|} \geqslant \varepsilon\right\} \\ &= \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |\mathcal{P}_r| \leqslant \frac{t(1-\varepsilon)}{1+\varepsilon}\right\}, \end{aligned}$$

for suitable value of $t = 3k^2$ and $\varepsilon = 0.5$, we get $K^n \subseteq \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{P}_r = k^2\right\} = \{1, 4, 9, 16, \ldots\}$.

Then, $\delta_{d,c}(K_\varepsilon^n) = \lim\limits_{n\to\infty} \frac{|K_\varepsilon^n|}{(c(n)-d(n))}$ is zero. Hence, $\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} 0$.

"Every convergent sequence is deferred statistically convergent". However, the converse is not always true.

Example 3.3. Let $(\mathbb{R}^m, \|\cdot\|)$ denote an FNS, and let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_m) \in \mathbb{R}^m$ be a "fixed non-zero vector". In this context, the "fuzzy norm" on $\mathbb{R}$ is specified as in Example 2.1, ensuring that $\|\mathcal{P}\|_C = \left(\mathcal{P}_1^2 + \cdots + \mathcal{P}_m^2\right)^{1/2}$ and $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1). Now define a sequence $(\mathcal{P}_r)$ in $\mathbb{R}$ as

$$\mathcal{P}_r = \begin{cases} sin\, n\pi, & r \neq n^2 \\ n, & \text{otherwise.} \end{cases}$$

where r belongs to $\mathbb{N}$.

Justification: For given ε, satisfying $0 < \varepsilon \leqslant b\left(\mathcal{P}_1^2 + \cdots + \mathcal{P}_m^2\right)^{1/2}$, it follows that

$$K^n(\varepsilon) = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \theta\|_0^+ \geqslant \varepsilon\right\} = \{1, 4, 9, 16, \ldots\}.$$

Therefore, the measure $\delta_{d,c}(K^n(\varepsilon))$ equals zero. Choosing $\varepsilon > b\left(\mathcal{P}_1^2 + \cdots + \mathcal{P}_m^2\right)^{1/2}$ results in $K^n(\varepsilon) = \emptyset$, and consequently, $\delta(\emptyset)$ is zero. Therefore, $\mathcal{P}_r$ converges in the sense of deferred statistical convergence. However, for the set $\{1, 4, 9, 16, \ldots\}$ contains infinitely many terms therefore the sequence $(\mathcal{P}_r)$ is not convergent.

The following summarizes some fundamental properties of statistical limits.

Theorem 3.1. Let $(\mathcal{P}_r)$ and $(\mathcal{Q}_r)$ be sequences in $(V, \|\cdot\|)$ and $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1), such that $\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} \mathcal{P}$ and $\mathcal{Q}_r \xrightarrow{\mathrm{st}_{d,c}(FN)} \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ belongs to V. Then we have

(i) $(\mathcal{P}_r + \mathcal{Q}_r) \overset{st_{d,c}(FN)}{\to} \mathcal{P} + \mathcal{Q}$,

(ii) $t\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} t\mathcal{P}(t$ belongs to $\mathbb{R})$,

Proof: (i) Assume that $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$ and $\mathcal{Q}_r \xrightarrow{st_{d,c}(FN)} \mathcal{Q}$. Since $\|\cdot\|_0^+$is a norm in the usual sense, we get

$$\|(\mathcal{P}_r + \mathcal{Q}_r) - (\mathcal{P} + \mathcal{Q})\|_0^+ \leqslant \|\mathcal{P}_r - \mathcal{P}\|_0^+ + \|\mathcal{Q}_r - \mathcal{Q}\|_0^+ \tag{2}$$

for all r belongs to $\mathbb{N}$. Now let $K(\varepsilon) = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|(\mathcal{P}_r + \mathcal{Q}_r) - (\mathcal{P} + \mathcal{Q})\|_0^+ \geqslant \varepsilon\right\}, K_1(\varepsilon) = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon/2\right\}$ and $K_2(\varepsilon) = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{Q}_r - \mathcal{Q}\|_0^+ \geqslant \varepsilon/2\right\}$. Then it follows from (2) that $K(\varepsilon) \subseteq K_1(\varepsilon) \cup K_2(\varepsilon)$. Then we have $\delta(K_1(\varepsilon)) = \delta(K_2(\varepsilon))$ is zero. Then we have $\delta(K(\varepsilon))$ is zero.

(ii) Let, $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$, then t belongs to $\mathbb{R} - \{0\}$ for every ε is greater than zero, we have

$$\lim_{n\to\infty} \frac{1}{(c(n) - d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{A}\left(\|\mathcal{P}_r - \mathcal{P}\|, \widetilde{0}\right) \geqslant \frac{\varepsilon}{|t|}\right\}\right|$$

is zero,

$$\Longrightarrow \lim_{n\to\infty} \frac{|t|}{(c(n) - d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{A}\left(\|\mathcal{P}_r - \mathcal{P}\|, \widetilde{0}\right) \geqslant |t|\frac{\varepsilon}{|t|}\right\}\right|$$

is zero,

$$\therefore \text{ it concluded that, } \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |c|\, \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}$$

$$\Longrightarrow \lim_{n\to\infty} \frac{1}{(c(n) - d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \mathcal{A}\left(\|t\mathcal{P}_r - t\mathcal{P}\|, \widetilde{0}\right) \geqslant \varepsilon\right\}\right|$$

is zero, $\Longrightarrow t\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} t\mathcal{P}$, ($t$ belongs to $\mathbb{R}$).

□

Theorem 3.2. Consider a sequence $(\mathcal{P}_r)$ in $(V, \|\cdot\|)$, and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). Then $(\mathcal{P}_r)$ is a "deferred statistical convergence sequence" in V iff $(\mathcal{P}_r)$ is a sequence for which $\exists$ a "convergent sequence" $(\mathcal{Q}_r)$ such that $\mathcal{P}_r = \mathcal{Q}_r$ for a.a.r (almost all r)

Proof: Let, $\{A_i : i \text{ belongs to } I\}$ be a "countable collection" of subsets of the natural numbers such that each A_i has density 1, then $\exists$ a subset A of the natural numbers with density 1, such that the set difference $A \setminus A_i$ is finite for every i belongs to I. Then we assume that $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$. For each $i \in \mathbb{N}$, let

$$A_i = \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \leqslant 1/i\right\}$$

and not that each set A_i has a density of 1 since the sequence $(\mathcal{P}_r)$ converges statistically in a deferred manner. Let A be defined as stated in the proof . It can be observed that $\mathcal{P}_r$ converges to $\mathcal{P}$ within A. In simpler terms, for any ε is greater than zero, there is any given ε, there's a specific index $N(\varepsilon)$ such that if r exceeds this index and is also an element of set A then the distance between $\mathcal{P}_r$ and $\mathcal{P}$ under a certain norm, denoted $|\cdot|_0^+$, is less than ε.

Now, let's construct a new sequence $(\mathcal{Q}_r)$ where $\mathcal{Q}_r = \mathcal{P}_r$ whenever n is an element of set A, and $\mathcal{Q}_r = \mathcal{P}$ whenever n is not in A. This construction illustrates that the sequence $(\mathcal{Q}_r)$ converges to $\mathcal{P}$ with respect to the "fuzzy norm" on V, ensuring that $\mathcal{P}_r$ is equal to $\mathcal{Q}_r$ for almost all terms.

Now assume that $\mathcal{P}_r = \mathcal{Q}_r$ for a.a.r (almost all r) and $\mathcal{Q}_r \xrightarrow{FN} \mathcal{P}$. Let ε is greater than zero. Then for each s belongs to $\mathbb{N} \cap [d(n)+1, c(n)]$ we can write

$$\left\{r \leqslant s : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\} \subseteq \{r \leqslant s : \mathcal{P}_r \neq \mathcal{Q}_r\} \cup \left\{r \leqslant s : \|\mathcal{Q}_r - \mathcal{P}\|_0^+ > \varepsilon\right\}. \tag{3}$$

The collection of integers in the second set on the right-hand side of equation (3) is characterized by a specific fixed count, which is denoted as $p = p(\varepsilon)$.

$$\text{Therefore,} \quad \lim_{s\to\infty} \frac{1}{s} \left|\left\{r \leqslant s : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}\right| \leqslant \lim_{s\to\infty} \frac{1}{s} \left|\{r \leqslant s : \mathcal{P}_r \neq \mathcal{Q}_r\}\right| + \lim_{s\to\infty} \frac{p}{s}$$

is zero,
since $\mathcal{P}_r = \mathcal{Q}_r$ for a.a.r. Hence $\|\mathcal{P}_r - \mathcal{P}\|_0^+ \geq \varepsilon$ for a.a.r, which means that $(\mathcal{P}_r)$ is "deferred statistically convergent". □

Definition 3.2. Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$ with $d = (d(n))$ and $c = (c(n))$ are sequences satisfying (1). We say that the sequence $(\mathcal{P}_r)$ in V is "deferred statistically Cauchy with respect to the fuzzy norm" on V if for every ε is greater than zero, $\exists$ $\mathbb{N}$, such that the density of $\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^+ \geqslant \varepsilon\right\}$ is zero, as $n \to \infty$.

Example 3.3. Consider the sequence $(\mathcal{P}_r)$ in the normed space $(\mathbb{R}, |\cdot|)$ given by $\mathcal{P}_r = \frac{1}{r}$. Let $d(n) = n$ and $c(n) = 2n$. These sequences $d(n)$ and $c(n)$ satisfy the conditions specified in Equation (1). Then it is show that the sequence $\mathcal{P}_r = \frac{1}{r}$ is "deferred statistically Cauchy with respect to the fuzzy norm" on V.

Theorem 3.3. Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$ with $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). Then every deferred statistically convergent sequence is also a "deferred statistically Cauchy sequence".

Proof: Let $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$ and ε is greater than zero. Then we have

$\lim\limits_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \frac{\varepsilon}{2}\right\}\right|$ is zero. Choose N belongs to $\mathbb{N}$ such that

$\lim\limits_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P} - \mathcal{P}_N\|_0^+ \geqslant \frac{\varepsilon}{2}\right\}\right|$ is zero.

$$
\begin{aligned}
\therefore \lim_{n\to\infty} \frac{1}{(c(n)-d(n))} &\left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^+ \geqslant \varepsilon\right\}\right| \\
&= \lim_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|(\mathcal{P}_r - \mathcal{P}) + (\mathcal{P} - \mathcal{P}_N)\|_0^+ \geqslant \varepsilon\right\}\right| \\
&\leq \lim_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \frac{\varepsilon}{2}\right\}\right| \\
&\quad + \lim_{n\to\infty} \frac{1}{(c(n)-d(n))} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P} - \mathcal{P}_N\|_0^+ \geqslant \frac{\varepsilon}{2}\right\}\right| \text{ is zero.}
\end{aligned}
$$

This shows $(\mathcal{P}_r)$ is "deferred statistically Cauchy". □

Theorem 3.4. Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1). Define $E_N(\varepsilon)$ as the set $\left\{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^+ \geqslant \varepsilon\right\}$ for any N belongs to $\mathbb{N}$. If $(\mathcal{P}_r)$ is deferred statistically Cauchy then for any ε is greater than zero, $\exists$ a set $A \subset \mathbb{N}$ and $\|\mathcal{P}_m - \mathcal{P}_r\|_0^+ < \varepsilon$ for all $m, r \notin A$ such that $\delta_{d,c}(A)$ is zero.

Definition 3.3. Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$, and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). An element $\mathcal{Q}$ belongs to V is considered a "deferred statistical limit point" of $(\mathcal{P}_r)$ if $\exists$ a subsequence of $(\mathcal{P}_r)$ that converges to $\mathcal{Q}$ outside a thin set. Formally, this means that $\exists$ a subsequence indexed by $A \subset \mathbb{N}$ such that $A = \{n_1 < n_2 < n_3 < \cdots < n_k < \dots\}$, where $\delta_{d,c}(A) = 0$ and $\lim\limits_{r\to\infty} \mathcal{P}_{r_k} = \mathcal{Q}$, and for any ε is greater than zero the measure of the set

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{n_k} - \mathcal{Q}\|_0^+ < \varepsilon\right\}$$

approaches zero as r tends to infinity. The set of all such "statistical limit points" of $(\mathcal{P}_r)$ is denoted by $\Lambda_{FN}(\mathcal{P}_r)$.

Remark 3.2. Suppose $(\mathcal{P}_r)$ is a sequence in V and $\left(\mathcal{P}_{r_j}\right)$ is a "subsequence" of $(\mathcal{P}_r)$. Define K as the set $\{r_j : j \in \mathbb{N}\}$. If $\delta(K)$ equals zero, we call $\left(\mathcal{P}_{r_j}\right)$ a "thin subsequence" of $(\mathcal{P}_r)$. Conversely, $\left(\mathcal{P}_{r_j}\right)$ is referred to as a "non-thin subsequence" if $\delta(K)$ is greater than zero or if $\delta(K)$ is undefined and $\bar{\delta}(K)$ is greater than zero [46].

Definition 3.4. Let $(\mathcal{P}_r)$ be a sequence in $(X, \|\cdot\|)$, and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). An element $\mathcal{Z}$ belongs to X is termed a "deferred statistical cluster point" of $(\mathcal{P}_r)$ if, for every ε is greater than zero, the measure $\bar{\delta}_{d,c}$ applied to the set

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_0^+ < \varepsilon\right\}$$

is greater than zero. The set of all "statistical cluster points" of $(\mathcal{P}_r)$ is represented by $\Gamma_{FN}(\mathcal{P}_r)$.

It is worth noting that being in $\Gamma_{FN}(\mathcal{P}_r)$ implies that $\bar{\delta}_{d,c}$ applied to the sets

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_\kappa^+ < \varepsilon\right\}$$

and

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_\kappa^- < \varepsilon\right\}$$

is greater than zero for every ε is greater than zero and each κ belongs to $[0,1]$.

Theorem 3.5. For any given sequence $(\mathcal{P}_r)$ in $(V, \|\cdot\|)$ and $d=(d(n))$ and $c=(c(n))$ be sequences satisfying (1), then we have

$$\Lambda_{FN}(\mathcal{P}_r) \subseteq \Gamma_{FN}(\mathcal{P}_r)$$

Proof: Suppose $\mathcal{Q}$ belongs to $\Lambda_{FN}(\mathcal{P}_r)$. Consequently, $\exists$ a "non-thin subsequence" $\left(\mathcal{P}_{r_j}\right)$ of $(\mathcal{P}_r)$ that converges to $\mathcal{Q}$, denoted by $\bar{\delta}_{d,c}(\{r_j : j \in \mathbb{N}\})$ is greater than zero. Since

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Q}\|_0^+ < \varepsilon\right\} \supseteq \left\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \left\|\mathcal{P}_{r_j} - \mathcal{Q}\right\|_0^+ < \varepsilon\right\}$$

for every ε is greater than zero, we have

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Q}\| + 0 < \varepsilon\} \supseteq \{r_j : j \in \mathbb{N}\} \setminus \{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{r_j} - \mathcal{Q}\| + 0 \geq \varepsilon\}$$

Since $\left(\mathcal{P}_{r_j}\right) \xrightarrow{st_{d,c}(FN)} \mathcal{Q}$, the set $\left\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \left\|\mathcal{P}_{r_j} - \mathcal{Q}\right\|_0^+ \geqslant \varepsilon\right\}$ is finite for any ε is greater than zero. Hence we have $\bar{\delta}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Q}\|_0^+ < \epsilon\}\right)$

$$\geq \bar{\delta}_{d,c}(\{r_j : j \in \mathbb{N}\}) - \bar{\delta}_{d,c}\left(\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{r_j} - \mathcal{Q}\|_0^+ \geq \epsilon\}\right) > 0$$

Thus, $\bar{\delta}_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Q}\|_0^+ < \varepsilon\right\}\right)$ is greater than zero for every ε is greater than zero, i.e., $\mathcal{Q}$ belongs to $\Gamma_{FN}(\mathcal{P}_r)$. □

Theorem 3.6. Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$ and $d=(d(n))$ and $c=(c(n))$ be sequences satisfying (1). If $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$, then

$$\Lambda_{FN}(\mathcal{P}_r) = \Gamma_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}.$$

Proof: Let $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$. Referring to Definitions 3.1 and 3.4, we conclude that $\mathcal{P}$ belongs to $\Gamma_{FN}(\mathcal{P}_k)$. If $\exists$ at least one $\mathcal{Z}$ belongs to $\Gamma_{FN}(\mathcal{P}_k)$ such that $\mathcal{Z} \neq \mathcal{P}$, we can infer the existence of $\varepsilon > 0$ such that

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\} \supseteq \left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_0^+ < \varepsilon\right\}$$

holds. Hence we get

$$\bar{\delta}_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}\right) \geqslant \bar{\delta}_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_0^+ < \varepsilon\right\}\right).$$

Since $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$, we have $\delta_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}\right)$ is zero, which indicates that

$$\bar{\delta}_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^+ \geqslant \varepsilon\right\}\right)$$

is zero.

Thus, we get

$$\bar{\delta}_{d,c}\left(\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_0^+ < \varepsilon\right\}\right)$$

is zero

a contradiction to z belongs to $\Gamma_{FN}(\mathcal{P}_r)$. Therefore, we should have $\Gamma_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}$.

Also, since $\mathcal{P}_r \xrightarrow{st(FN)} \mathcal{P}$. By Theorem 3.2 and Definition 3.3, we conclude that $\mathcal{P} \in \Lambda_{FN}(\mathcal{P}_r)$. Subsequently, Theorem 3.5 provides further insights $\Lambda_{FN}(\mathcal{P}_r) = \Gamma_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}$. □

4|Conclusion

In this paper, we introduced the concepts of "deferred statistically convergent" sequences and "deferred statistically Cauchy" sequences within a "fuzzy normed linear space" (FNS), establishing fundamental results and examining their basic properties. We defined "deferred statistical limit points" for sequences in FNS and investigated their relationships and implications. We also defined "deferred statistical cluster points" for sequences in FNS and examined their implications. Future research could extend these concepts to "intuitionistic fuzzy normed spaces".

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Data Availability

"All data supporting the reported findings in this research paper are provided within the manuscript".

Conflicts of Interest

The authors declare that they have no conflict of interest.

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